SUBJECT 104: SURVIVAL MODELS

Aim
The aim of the Survival Models course is to provide grounding in survival models and their simple applications.

Objectives
Note that "life" in the context of this Syllabus does not refer only to human life, but includes, where appropriate, situations subject to other forms of decrement, for example withdrawal rates under life insurance contracts.
On completion of the course the trainee actuary will be able to:

(i) Explain the concept of survival models.
1. Describe the model of lifetime or failure time from age x as a random variable.
2. State the consistency condition between the random variable representing lifetimes from different ages.
3. Define the distribution and density functions of the random future lifetime, the survival function, the force of mortality or hazard rate, and derive relationships between them.
4. Define the actuarial symbols ${}_t p_x$ and ${}_t q_x$, and derive integral formulae for them.
5. State the Gompertz and Makeham laws of mortality.
6. Define the curtate future lifetime from age x and state its probability function.
7. Define the expected value and variance of the complete and curtate expected future lifetimes and derive expressions for them. Define the symbols e_x and e_x and derive an approximate relation between them.
8. Describe the two-state model of a single decrement and compare its assumptions with those of the random lifetime model.

(ii) Describe estimation procedures for lifetime distributions.
1. Describe the various ways in which lifetime data might be censored.
2. Describe the estimation of the empirical survival function in the absence of censoring, and what problems are introduced by censoring.
3. Describe the Kaplan-Meier (or product limit) estimate of the survival function in the presence of censoring, explain how it arises as a maximum likelihood estimate, compute it from typical data and estimate its variance.
4. Describe the Nelson-Aalen estimate of the cumulative hazard rate in the presence of censoring, explain how it arises as a maximum likelihood estimate, and compute it from typical data and estimate its variance.
5. Describe the Cox model for proportional hazards, derive the partial likelihood estimate in the absence of ties, and state its asymptotic distribution.

(iii) Describe statistical models of transfers between multiple states, including processes with single or multiple decrements, and derive relationships between probabilities of transfer and transition intensities.
1. State the assumptions underlying Markov models of transfers between a finite number of states in continuous time, in terms of given time-varying transition intensities.
2. Define ${}_t p_x^{gh}$, ${}_t p_x^{\overline{gg}}$, ${}_t \mu_x^{gh}$ and express ${}_t \mu_x^{gh}$ in terms of ${}_t p_x^{gh}$
3. Derive and solve equations for $\frac{\partial}{\partial t} {}_t p_x^{\overline{gg}}$

4. Derive equations for $\frac{\partial}{\partial t}\,{}_t p_x^{gh}$ and solve them for simple models.

(iv) Derive maximum likelihood estimators for the transition intensities in models of transfers between states with piecewise constant transition intensities.

1. Describe an observational plan in respect of a finite number of individuals observed during a finite period of time, and define the resulting statistics, including the waiting times.
2. Derive the likelihood function for constant transition intensities in a Markov model of transfers between states given the statistics in 1.
3. Derive maximum likelihood estimators for the transition intensities in 2 and state their asymptotic joint distribution.
4. Describe the Poisson approximation to the estimator in 3 in the case of a single decrement and its advantages and disadvantages.

(v) Describe the Binomial model of mortality, derive a maximum likelihood estimator for the probability of death and compare the Binomial model with the multiple state models.

1. Describe the Binomial model of the mortality of a group of identical individuals subject to no other decrements between two given ages.
2. Derive the maximum likelihood estimator for the rate of mortality in the Binomial model and its mean and variance.
3. Describe the advantages and disadvantages of the multiple state model and the Binomial model, including consistency, efficiency, simplicity of the estimators and their distributions, application to practical observational plans and generality.

vi) Describe how to estimate transition intensities depending on age, exactly or using the census approximation.

1. Explain the importance of dividing the data into homogeneous classes, including subdivision by age and sex.
2. Describe the principle of correspondence and explain its fundamental importance in the estimation procedure.
3. Specify the data needed for the exact calculation of a central exposed to risk (waiting time) depending on age and sex.
4. Calculate a central exposed to risk given the data in 3.
5. Explain how to obtain estimates of transition probabilities, including in the single decrement model the actuarial estimate based on the simple adjustment to the central exposed to risk.
6. Explain the assumptions underlying the census approximation of waiting times.
7. Explain the concept of rate interval.
8. Develop census formulae given the following definitions of age:
   - age at birthday
   - age at specified calendar date
   - age at a specified policy anniversary

where the age may be classified as next, last, or nearest relative to the birthday, calendar date, or policy anniversary as appropriate.
The deaths and census data may use different definitions of age.
9. Specify the age to which estimates of transition intensities or probabilities apply in the cases in 8.

(vii) Describe how to test crude estimates for consistency with a standard table or a set of graduated estimates, and describe the process of graduation.

1. Describe the following statistical tests of crude estimates, for comparison with a standard table:

- chi-square test
- standardised deviations test
- sign test
- cumulative deviation test
- grouping of signs test
- serial correlations test

For each test describe:

- the formulation of the hypothesis
- the test statistic
- the distribution of the test statistic using approximations where appropriate
- the application of the test statistic

2. Describe the reasons for graduating crude estimates of transition intensities or probabilities, and state the desirable properties of a set of graduated estimates.
3. Describe a test for smoothness of a set of graduated estimates.
4. Describe the process of graduation by the following methods, and state the advantages and disadvantages of each:

- parametric formula
- standard table
- graphical

5. Describe how the tests in 1 should be amended to compare crude and graduated sets of estimates.
6. Describe how the tests in 1 should be amended to allow for the presence of duplicate policies.
7. Carry out a comparison of a set of crude estimates and a standard table, or of a set of crude estimates and a set of graduated estimates.

(viii) Define simple assurance and annuity contracts, and develop formulae for the means and variances of the present values of the payments under these contracts, assuming constant deterministic interest.

1. Define the following terms:

- Whole life assurance
- Term assurance
- Pure endowment assurance
- Endowment assurance
- Whole life level annuity
- Temporary level annuity
- Premium
- Benefit

including assurance and annuity contracts where the benefits are deferred.
2. Define the following probabilities: ${}_{n|m}q_x$, ${}_{n|}q_x$.
3. Obtain expressions in the form of sums for the mean and variance of the present value of benefit payments under each contract above, in terms of the curtate random future lifetime, assuming that death benefits are payable at the end of the year of death and that annuities are paid annually in advance or in arrears, and, where appropriate, simplify these expressions into a form suitable for evaluation by table look-up or other means.
4. Obtain expressions in the form of integrals for the mean and variance of the present value of benefit payments under each contract above, in terms of the random future lifetime, assuming that death benefits are payable at the moment of death and that annuities are paid continuously, and, where appropriate, simplify these expressions into a form suitable for evaluation by table look-up or other means.

5. Define the symbols A_x, $A_{x:\overline{n}|}$, $A^1_{x:\overline{n}|}$, a_x, $a_{x:\overline{n}|}$, ${}_{m|}a_{x:\overline{n}|}$, and their continuous equivalents.

6. Derive the relations $A_x = 1 - d\,\ddot{a}_x$, $A_{x:\overline{n}|} = 1 - d\,\ddot{a}_{x:\overline{n}|}$, and their continuous equivalents.

7. Derive relations between level annual annuities payable in advance and in arrears, and between level temporary, deferred and whole life annuities.

8. Define the expected accumulation of the benefits in 1, and obtain expressions for them corresponding to the expected present values in 3 and 4.

(ix) Compute expected present values and variances of simple benefits by table look-up or by using life tables.

1. Describe the life table functions l_x and d_x.

2. Express the following life table probabilities in terms of the functions in 1: ${}_np_x$, ${}_nq_x$, ${}_{n|m}q_x$.

3. Express the expected values and variances in objective (viii) 3 in terms of the functions in 1.

4. Evaluate the expected values and variances in objective (viii) 3 by table look-up, where appropriate, including the use of the relationships in objectives (viii) 6 and 7.

5. Define commutation functions D_x, N_x, C_x and M_x, and evaluate the expected values and variances in objective (viii) 3 using them.

6. Derive approximations for, and hence evaluate, the expected values and variances in objective (viii) 4 in terms of those in objective (viii) 3.

7. Evaluate the expected accumulations in objective (viii) 8.

(x) Describe and calculate net premiums and net premium policy values of simple assurance and annuity contracts.

1. Define the random future loss under an assurance or annuity contract, and state the principle of equivalence.

2. Define and calculate:
   - net single premiums for the assurance and annuity benefits in objective (viii) 1.
   - net level annual premiums for the assurance and deferred annuity contracts in objective (viii) 1.
   - net level premiums payable continuously for the assurance and deferred annuity contracts in objective (viii) 1.

3. State why a life insurance company will set up reserves.

4. Describe prospective and retrospective policy values.

5. Define and evaluate prospective and retrospective net premium policy values in respect of the contracts in objective (viii) 1, with premiums as in 2.

6. Derive recursive relationships between net premium policy values at annual intervals, for contracts with death benefits paid at the end of the year of death, and annual premiums.

7. Define and calculate, for a single policy or a portfolio of policies (as appropriate):
   - Death strain at risk
   - Expected Death strain
   - Actual Death strain
   - Mortality profit

8. Derive Thiele's differential equation, satisfied by net premium policy values for contracts with death benefits paid at the moment of death, and premiums payable continuously.

9. Show that prospective and retrospective net premium policy values are equal when calculated on the same basis.

The Main Reading: UK Institute of Actuaries Core Reading for subject 104 Survival Models

Additional Reading:

1. Benjamin, Bernard and Pollard, John H., *The analysis of mortality and other actuarial statistics.* 3rd ed. Institute of Actuaries and Faculty of Actuaries, 1993 ASIN: 0750608501
2. Dick London, *Survival Models and Their Estimation*, 3rd Edition, ACTEX, 1997 ISBN 1566982685
3. Dick London, *Survival Models and Their Estimation Manual*, 3rd Edition, ACTEX, 1997 ISBN 1566982928
4. Haberman, S. and Pitacco, E., *Actuarial models for disability insurance* - Chapman & Hall, 1999 ISBN: 0849303893
5. Elandt-Johnson, Regina C. and Johnson, Norman L., *Survival models and data analysis* – 2nd ed. John Wiley, 1999 ISBN: 0471349925
6. Klein J. and Moeschberger M., *Survival Analysis: techniques for censored and truncated data.* Springer, 1997 ISBN 038795399X